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# Remarks on the state convergence of nonlinear systems given any $L^p$ input

Bayu Jayawardhana

**Abstract**—In this paper, we study the state convergence of  $L^p$ -stable systems and input-to-state stable (ISS) systems given any  $L^p$  input signal where  $p \in [1, \infty)$ . Under some conditions on the system equations, it is shown that if the system is dissipative with a proper storage function  $H$  and supply rate  $s(y, u) = \|u\|^p - k\|y\|^p$ ,  $k > 0$  and it is zero-state detectable then the state  $x$  of the system converges to zero for any  $L^p$  input. This implies also that the system is also  $L^p$ -stable.

Using the same technique, if the system is ISS, under some conditions on the system equations and on the smooth ISS Lyapunov function, then the state  $x$  of the system converges to zero for any  $L^p$  input.

## I. INTRODUCTION

Consider the system  $\mathbf{P}$  described by

$$\dot{x} = f(x, u), \quad (1)$$

$$y = h(x), \quad (2)$$

where the state  $x$ , the input  $u$  and the output  $y$  are functions of  $t \geq 0$ , such that  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  and  $y(t) \in \mathbb{R}^l$ . We assume that  $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$  with  $f(0, u) = 0 \Leftrightarrow u = 0$  and  $h \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^l)$  with  $h(0) = 0$ .

The convergence of the state trajectory  $x$  of  $\mathbf{P}$  given a converging input signal  $u$  (i.e.,  $\|u(t)\| \rightarrow 0$  as  $t \rightarrow \infty$ ) for nonlinear systems has been studied by Sontag [13]. Suppose that 0 is the global asymptotic stable (GAS) equilibrium point of  $\mathbf{P}$  (taking  $u = 0$ ). It is shown in [13] that if for a given converging input  $u$  and initial state  $x(0)$ , there exists a unique solution  $x(t)$  of (1) defined for all  $t \geq 0$  and  $x$  is bounded, then  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This result is generalized by Ryan [8] for  $L^p$  input. Using the same assumption on the global asymptotic stability of the origin and assuming that for all compact set  $K \in \mathbb{R}^n$  there exists  $c > 0$  such that

$$\|f(x, u) - f(x, 0)\| \leq c\|u\| \quad \forall u \in \mathbb{R}^m, x \in K, \quad (3)$$

it is shown in [8] that if for a given  $L^p$  input  $u$  and initial state  $x(0)$ , there exists a solution of the state trajectory  $x$  defined for all  $t \geq 0$  and  $x$  is bounded, then  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Let us recall again the definition of  $L^p$  stable systems and input-to-state stable systems. The system  $\mathbf{P}$  is  $L^p$  stable if for any  $L^p$  input  $u$  and initial state  $x(0)$ , there exists a unique solution  $x(t)$  of (1) for all  $t \geq 0$  and the output  $y$  is in  $L^p$  (see also van der Schaft [9] for details). The system  $\mathbf{P}$  is input-to-state stable (ISS) if for any  $L^\infty$  input and initial state  $x(0)$ , there exists a unique solution  $x(t)$  of (1) for all  $t \geq 0$  and

$x$  is bounded and satisfies  $\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|u\|_{L^\infty})$  where  $\beta: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{KL}$  and  $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{K}$  (the definition of these notions are given in Section II, see also Sontag and Wang [12] for details).

In this note, we show the convergence of the state of nonlinear systems given an  $L^p$  input signal where  $p \in [1, \infty)$ . Under some conditions on  $f$  in (1), it is shown that if the system  $\mathbf{P}$  is dissipative with respect to supply rate  $s(y, u) = \|u\|^p - k\|y\|^p$ ,  $k > 0$  and with a proper storage function  $H$  (the definition of dissipativity and properness is given in Section II) and it is zero-state detectable, then for any  $L^p$  input  $u$  there exists a unique solution  $x(t)$  of (1) for all  $t \geq 0$  and the state  $x$  converges to zero. This results implies also that the systems satisfying these conditions are  $L^p$ -stable.

For an ISS system  $\mathbf{P}$ , we show that if the smooth ISS Lyapunov function  $V(x)$  satisfies  $\dot{V} \leq -\alpha_3(\|x(t)\|) + k\|u(t)\|^p$  where  $\alpha_3 \in \mathcal{K}$  and  $k > 0$ , then for any  $L^p$  input  $u$  there exists solutions  $x$  of (1) for all  $t \geq 0$  and the state  $x$  converges to zero.

We extend the result of [13] and [8] in the following way: we use a technique from infinite-dimensional linear system theory to show that for an  $L^p$  input there exists a unique solution of  $x$  for all  $t \geq 0$  and we show that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  using an infinite-dimensional version of the La Salle invariance principle. Here, we allow the function  $f$  to satisfy a weaker condition than (3) (the Lipschitz condition assumed in [8]). For example, we allow  $f(x, u) = -x(1 + |u|^{\frac{1}{2}}) + u$  which satisfy the following condition: for each compact set  $K \in \mathbb{R}^n$  there exist  $c_1, c_2 > 0$  such that

$$\begin{aligned} \|f(x, u) - f(x, 0)\| &= \|-x(1 + |u|^{\frac{1}{2}}) + u + x\| \\ &\leq c_1 + c_2\|u\| \quad \forall x \in K, u \in \mathbb{R}^m. \end{aligned}$$

The result of this paper reveals an additional property that an  $L^p$ -stable system can have. The standard properties of an  $L^p$ -stable system are:

- 1) If the input is zero and the system satisfy a detectability condition, then the state converges to zero (see [9]).
- 2) Using a Bärbalat's type argument, it can be shown that if the storage function is proper then the output converges to zero for any  $L^p$  input signal (see [14]).

However, these standard stability results do not describe the convergence of the state trajectory to zero for any  $L^p$  input. Intuitively, if  $u \in L^p$ , then for very large  $\tau$  the energy left in  $u$  for  $t \geq \tau$  becomes negligible, and the system behaves as it would for  $u = 0$ , i.e., we have  $x(t) \rightarrow 0$ . However, a rigorous proof of this result is not easy. Our proof uses techniques from infinite-dimensional system theory.

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This paper is an extension of the results by Jayawardhana and Weiss [3] and by Jayawardhana [4] where the convergence of the state of strictly output passive systems given any  $L^2$  input is shown. The paper [3] proposes an LTI controller for passive nonlinear plants to solve the disturbance rejection problem where the disturbance signal can be decomposed into a component generated by an exosystem and an  $L^2$  signal.

For a stable LTI system, the state convergence given an  $L^p$  input signal where  $p \in [1, \infty)$  can be shown easily. Indeed, suppose that  $\mathbf{P}$  is described by

$$\dot{x} = Ax + Bu, \quad (4)$$

$$y = Cx + Du, \quad (5)$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  and  $y(t) \in \mathbb{R}^l$ . From the detectability and the dissipativity of  $\mathbf{P}$  or from the input-to-state stability of  $\mathbf{P}$ , it follows that  $A$  is Hurwitz. Thus,  $u \in L^p$  implies that  $x \in L^p$ . From (4) we also have  $\dot{x} \in L^p$ . Thus, using generalized Bärbalat's lemma (see for example [7])  $x, \dot{x} \in L^p \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0$ .

## II. PRELIMINARIES

**Notation.** Throughout this paper,  $\mathbb{R}_+ = [0, \infty)$ . For a finite-dimensional vector  $x$ , we use the norm  $\|x\| = (\sum_n |x_n|^2)^{\frac{1}{2}}$ . For any finite-dimensional vector space  $\mathcal{V}$  endowed with a norm  $\|\cdot\|$ , the space  $L^p(\mathbb{R}_+, \mathcal{V})$ ,  $p \in [1, \infty)$ , consists of all the measurable functions  $f : \mathbb{R}_+ \rightarrow \mathcal{V}$  such that  $\int_0^\infty \|f(t)\|^p dt < \infty$ . The  $L^p$ -norm of a function  $f \in L^p(\mathbb{R}_+, \mathcal{V})$  is given by  $\|f\|_{L^p} = (\int_0^\infty \|f(t)\|^p dt)^{\frac{1}{p}}$ . For  $f \in L^p(\mathbb{R}_+, \mathcal{V})$  and  $T > 0$ , we denote by  $f_T$  the truncation of  $f$  to  $[0, T]$ . The space  $L^p_{loc}(\mathbb{R}_+, \mathcal{V})$  consists of all the measurable functions  $f : \mathbb{R}_+ \rightarrow \mathcal{V}$  such that  $f_T \in L^p(\mathbb{R}_+, \mathcal{V})$ , for all  $T > 0$ . The space  $\mathcal{W}^{1,p}(\mathbb{R}_+, \mathcal{V})$  consists of all the functions  $f : \mathbb{R}_+ \rightarrow \mathcal{V}$  such that  $f, \frac{df}{dt} \in L^p(\mathbb{R}_+, \mathcal{V})$  (where  $\frac{df}{dt}$  is understood in the sense of distributions). The space  $\mathcal{C}(\mathbb{R}^l, \mathbb{R}^p)$  (respectively  $\mathcal{C}^1(\mathbb{R}^l, \mathbb{R}^p)$ ) consists of all the continuous (respectively continuously differentiable) functions  $f : \mathbb{R}^l \rightarrow \mathbb{R}^p$ . For any  $\varepsilon \geq 0$ , we denote  $\mathbf{B}_\varepsilon = \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}$ .

The set  $\mathcal{K}$  consists of all the continuous functions  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\alpha(0) = 0$  and it is strictly increasing. The set  $\mathcal{K}_\infty$  consists of all the functions  $\alpha \in \mathcal{K}$  such that  $\alpha(s) \rightarrow \infty$  as  $s \rightarrow \infty$ . The set  $\mathcal{KL}$  consists of all the functions  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that  $\beta(\cdot, t) \in \mathcal{K}$  and  $\beta(s, \cdot)$  converges to zero.  $H : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called *proper* if for all  $c \geq 0$ , the set  $\{x \mid H(x) \leq c\}$  is compact or it is equivalent to say that  $H(x) \rightarrow \infty$  whenever  $\|x\| \rightarrow \infty$ .

We consider the system  $\mathbf{P}$  described by (1), (2) satisfying the assumptions stated thereafter.

We recall a result on the existence and uniqueness of the solution of the differential equation (1) (see also Sontag [11, Appendix C] for details).

**Definition 2.1:** A solution of (1) with a measurable input  $u$  on an interval  $\mathcal{I}$  containing 0 is an absolutely continuous function  $x : \mathcal{I} \rightarrow \mathbb{R}^n$  such that

$$x(t) - x(0) = \int_0^t f(x(\tau), u(\tau)) d\tau \quad \forall t \in \mathcal{I}.$$

**Theorem 2.2:** Assume that  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  is measurable,  $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$  and the following two conditions hold for every  $a \in \mathbb{R}^n$ :

- (S1) There exists a constant  $c > 0$  and a locally integrable function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$\|f(x, u(t)) - f(y, u(t))\| \leq \alpha(t) \|x - y\|$$

for almost every  $t \in \mathbb{R}_+$  and for all  $x, y \in a + \mathbf{B}_c$ .

- (S2) There exists a locally integrable function  $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$\|f(a, u(t))\| \leq \beta(t)$$

for almost every  $t \in \mathbb{R}_+$ .

Then for every  $x(0) \in \mathbb{R}^n$  there exists  $I(x(0)) > 0$  and a unique solution of (1) with input  $u$  on  $[0, I(x(0)))$ .

This theorem is an immediate consequence of [11, Theorem 36]. We need this result in the next sections when dealing with an  $L^p$  input signal.

**Corollary 2.3:** Suppose that  $u$  and  $f$  are as in Theorem 2.2 and  $[0, I(x(0)))$  (where  $I(x(0)) > 0$ ) is the maximal interval of existence of the solution of (1). If  $I(x(0)) < \infty$  then for every compact set  $K \subset \mathbb{R}^n$ , there exists  $T \in [0, I(x(0)))$  such that  $x(T) \notin K$ .

*Proof:* The property (S1) and (S2) in Theorem (2.2) implies also that for any compact  $K \subset \mathbb{R}^n$ , there is a locally integrable function  $\gamma$  such that

$$\|f(x, u(t))\| \leq \gamma(t), \quad (6)$$

for almost every  $t \in \mathbb{R}_+$  and for all  $x \in \mathcal{K}$ . Indeed, given any  $a \in K$ , there exists  $c > 0$  and function  $\alpha$  and  $\beta$  as in the Theorem 2.2. Thus,

$$\|f(x, t)\| \leq \|f(a, t)\| + \|f(x, t) - f(a, t)\| \leq \beta(t) + c\alpha(t),$$

for all  $x \in a + \mathbf{B}_c$  and almost every  $t \in \mathbb{R}_+$ . Denote the last inequality above by  $\gamma_a(t) = \beta(t) + c\alpha(t)$  which is locally integrable. Consider the open covering of  $K$  by the sets of the form  $\mathbf{B}_{c_j} + a_j$ ,  $a_j \in K$ ,  $j = \{1, 2, \dots\}$ . By compactness, the open covering has a finite subcovering, i.e.,  $j$  is finite. Choose  $\gamma(t) = \max_j \{\gamma_{a_j}(t)\}$ , then  $\gamma$  satisfies (6) since  $\gamma_j$  is locally integrable for each  $j$ .

We prove the corollary by using contradiction. Suppose that there exists a compact set  $K \subset \mathbb{R}^n$  such that  $x(t) \in K$  for all  $t \in [0, I(x(0)))$ . First, we show that  $\lim_{t \rightarrow I(x(0))} x(t)$  exists. For the compact set  $K$ , we know that there exists a locally integrable function  $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that (6) holds. Then we have

$$\|x(t_k) - x(t_j)\| \leq \int_{t_j}^{t_k} \|f(x(\tau), u(\tau))\| d\tau \leq \int_{t_j}^{t_k} \gamma(\tau) d\tau,$$

where  $t_k, t_j \in [0, I(x(0)))$ . Since  $\|x(t_k) - x(t_j)\| \rightarrow 0$  as  $t_k, t_j \rightarrow I(x(0))$ . Since  $K$  is a complete metric space,  $\lim_{t \rightarrow I(x(0))} x(t)$  exists and  $x(I(x(0))) \in K$ . However, we could use again Theorem 2.2 with  $I(x(0))$  as the initial time and  $x(I(x(0)))$  as the initial state to show the existence of solution of (1) on

the interval  $[I(x(0)), \eta)$ ,  $\eta > I(x(0))$ . This shows that  $I(x(0))$  is not the maximal interval of existence of the solution of (1).  $\square$

If  $I(x(0)) < \infty$  is as in Corollary 2.3, then it is called the *finite escape time*.

Let  $\mathcal{X}$  be a metric space with distance  $\mu$ . A set  $G \subset \mathcal{X}$  is *relatively compact* if the closure of  $G$  is compact. Let  $z: \mathbb{R}_+ \rightarrow \mathcal{X}$ . A point  $\xi \in \mathcal{X}$  is said to be an  $\omega$ -limit point of  $z$  if there exists a sequence  $(t_n)$  in  $\mathbb{R}_+$  such that  $t_n \rightarrow \infty$  and  $z(t_n) \rightarrow \xi$ . The set of all the  $\omega$ -limit points of  $z$  is denoted by  $\Omega(z)$ .

A map  $\pi: \mathbb{R}_+ \times \mathcal{X} \rightarrow \mathcal{X}$  is said to be a *semiflow on  $\mathcal{X}$*  if  $\pi$  is continuous,  $\pi(0, x_0) = x_0$  for all  $x_0 \in \mathcal{X}$  and

$$\pi(s+t, x_0) = \pi(s, \pi(t, x_0)) \quad \forall s, t \in \mathbb{R}_+ \quad \forall x_0 \in \mathcal{X}.$$

A non-empty set  $G \subset \mathcal{X}$  is  $\pi$ -invariant if  $\pi(t, G) = G$  for all  $t \in \mathbb{R}_+$ .

**Proposition 2.4:** Let  $\pi: \mathbb{R}_+ \times \mathcal{X} \rightarrow \mathcal{X}$  be a semiflow on a metric space  $\mathcal{X}$ . Let  $x_0 \in \mathcal{X}$  and denote  $z(t) = \pi(t, x_0)$ . If  $z(\mathbb{R}_+)$  is relatively compact, then  $\Omega(z)$  is non-empty, compact,  $\pi$ -invariant and

$$\lim_{t \rightarrow \infty} \mu(z(t), \Omega(z)) = 0. \quad (7)$$

The proof is a straightforward extension from the result for finite-dimensional systems where  $\mathcal{X} \subset \mathbb{R}^n$  (see, for example, the result by La Salle [6] or by Logemann and Ryan [7]). Several extension of the La Salle invariance principle to the infinite-dimensional systems can also be found in Hale [1] and Slemrod [10]. This result will be used for an infinite-dimensional system in Section III and IV. The proof is given below to make the paper self-contained. We mention that  $\Omega(z)$  is also connected.

*Proof:* Since  $z(\mathbb{R}_+)$  is relatively compact,  $\Omega(z)$  is non-empty and compact.

To prove  $\pi$ -invariance, take  $\xi \in \Omega(z)$ , so that there exists a sequence  $(t_n)$  in  $\mathbb{R}_+$  such that  $t_n \rightarrow \infty$  and  $z(t_n) \rightarrow \xi$ . Take  $t > 0$ , then

$$\pi(t, \xi) = \lim_{n \rightarrow \infty} \pi(t, z(t_n)) = \lim_{n \rightarrow \infty} \pi(t + t_n, x_0) \in \Omega(z),$$

so that  $\pi(t, \Omega(z)) \subset \Omega(z)$ . To prove the opposite inclusion, take  $\eta \in \Omega(z)$ , so that  $\eta = \lim_{n \rightarrow \infty} z(\tau_n)$  for some sequence  $(\tau_n)$  with  $\tau_n \rightarrow \infty$ . The sequence  $\pi(\tau_n - t, x_0)$  (defined for  $n$  large enough, so that  $\tau_n - t > 0$ ) being contained in a compact set, has a convergent subsequence  $\pi(\theta_n, x_0)$ , where  $(\theta_n)$  is a subsequence of  $(\tau_n - t)$ . If we put  $\xi = \lim_{n \rightarrow \infty} \pi(\theta_n, x_0)$ , then  $\pi(t, \xi) = \eta$ .

To prove (7), assume that (7) is false. Then there exists a sequence  $(t_n) \in \mathbb{R}_+$  such that  $t_n \rightarrow \infty$  and  $\mu(z(t_n), \Omega(z)) \geq \varepsilon > 0$  for all  $n$ . This is a contradiction since for a subsequence  $(\theta_n)$  of  $(t_n)$ , we have  $z(\theta_n) \rightarrow \xi \in \Omega(z)$ .  $\square$

### III. STATE CONVERGENCE FOR AN $L^p$ STABLE SYSTEM GIVEN ANY $L^p$ INPUT

We need a few more definitions for this section. The system  $\mathbf{P}$  is called *dissipative with respect to supply rate*

$s(y, u) = \|u\|^p - k\|y\|^p$ ,  $k > 0$ ,  $p \in [1, \infty)$ , if there exists a storage function  $H \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$  such that

$$\frac{\partial H(x)}{\partial x} f(x, u) \leq \|u\|^p - k\|y\|^p. \quad (8)$$

See also Willems [17] for description on dissipative systems.

$\mathbf{P}$  is said to be *zero-state detectable* if the following is true: If  $u(t) = 0$  and  $x$  is a unique solution of (1) on  $\mathbb{R}_+$  such that  $y(t) = 0$  for all  $t \geq 0$ , then  $\lim_{t \rightarrow \infty} x(t) = 0$  (see also [2, Definition 10.7.3]).

Suppose that  $p \in [1, \infty)$ . We need additional assumptions on the function  $f$  in (1):

**(A1)** For every compact set  $K \subset \mathbb{R}^n$ , there exist constants  $c_1, c_2 > 0$  such that

$$\|f(a, u) - f(b, u)\| \leq (c_1 + c_2\|u\|^p)\|a - b\|, \quad (9)$$

for all  $u \in \mathbb{R}^m$ ,  $a, b \in K$ .

**(A2)** For each fixed  $a \in \mathbb{R}^n$ , there exist constants  $c_3, c_4 > 0$  such that

$$\|f(a, u)\| \leq c_3 + c_4\|u\|^p \quad \forall u \in \mathbb{R}^m. \quad (10)$$

**Remark 3.1:** For any  $p \in [1, \infty)$ , it can be shown that the Assumptions **(A1)** and **(A2)** are satisfied for affine passive nonlinear systems  $\mathbf{P}$  described by

$$\dot{x} = \tilde{f}(x) + g(x)u, \quad (11)$$

$$y = h(x), \quad (12)$$

where  $\tilde{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ ,  $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^{n \times m})$ ,  $g(0)$  has rank  $m$  and  $h$  is as in (2). This class of systems includes also the port-controlled Hamiltonian systems [9].

For any  $\tau \geq 0$ , we denote by  $\mathbf{S}_\tau^*$  the left-shift operator by  $\tau$ , acting on  $X = L^p(\mathbb{R}_+, \mathbb{R}^m)$ . Suppose that  $d_0 \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  and  $d_t = \mathbf{S}_t^* d_0$ , it follows that  $d_t \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  for all  $t \geq 0$  and the following equation holds for almost every  $t \geq 0$ :

$$\frac{d}{dt} \|d_t\|_{L^p}^p = \frac{d}{dt} \int_t^\infty \|d_0(\xi)\|^p d\xi = -\|d_0(t)\|^p. \quad (13)$$

**Theorem 3.2:** Suppose that  $p \in [1, \infty)$  and let the system  $\mathbf{P}$  defined by (1), (2) be zero-state detectable and satisfies **(A1)** and **(A2)**. Assume that  $\mathbf{P}$  has a storage function  $H$  such that it is dissipative with respect to supply rate  $s(y, u) = \|u\|^p - k\|y\|^p$ ,  $k > 0$ . Suppose that  $H(x) > 0$  for  $x \neq 0$ ,  $H(0) = 0$  and  $H$  is proper.

Then for every initial condition  $x(0) \in \mathbb{R}^n$  and for every  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ , there exists a unique solution  $x$  of (1) on  $\mathbb{R}_+$  and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  (and hence  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ ).

*Proof:* Using **(A1)**, we have that for every compact set  $K \subset \mathbb{R}^n$ , there exist constants  $c_1, c_2 > 0$  such that (9) holds. By denoting  $\alpha(t) = c_1 + c_2\|u(t)\|^p$  and since  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ , it is easy to see that  $\alpha$  is locally integrable. Hence, the condition **(S1)** in Theorem 2.2 holds for the state equation (1).

Using the assumption **(A2)**, we have that for each fixed  $a \in \mathbb{R}^n \times \mathbb{R}^l$ , there exist constants  $c_3, c_4 > 0$  such that (10) holds.

By denoting  $\beta(t) = c_3 + c_4 \|u(t)\|^p$  and since  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ ,  $\beta$  is locally integrable and satisfies the condition (S2) in Theorem 2.2 for the state equation (1).

Then using  $\alpha, \beta$  as above and using initial value  $x(0) \in \mathbb{R}^n$ , it follows from Theorem 2.2 that there exists  $I(x(0)) > 0$  and a unique solution of (14) with input  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  on  $\mathcal{J} = [0, I(x(0))]$ . In particular,  $x$  is absolutely continuous as a function of  $t$  on  $\mathcal{J}$ .

We define an infinite-dimensional signal generator for the signal  $u$ . This signal generator has the state space  $X = L^p(\mathbb{R}_+, \mathbb{R}^m)$  and the evolution of its state is governed by the operator semigroup  $(S_t^*)_{t \geq 0}$ . Thus, the state of the signal generator at time  $t$  is  $d_t = S_t^* d_0$ , where  $d_0 \in X$  is the initial state. The generator of this semigroup is  $\mathcal{A} = \frac{d}{d\xi}$  with domain  $\mathcal{D}(\mathcal{A}) = \mathcal{W}^{1,p}(\mathbb{R}_+, \mathbb{R}^m)$ . The observation operator of this signal generator is  $\mathcal{C}$ , defined for  $\phi \in \mathcal{D}(\mathcal{A})$  by  $\mathcal{C}\phi = \phi(0)$ . It can be checked that  $\mathcal{C}$  is admissible in the sense of Weiss [16]. We need the Lebesgue extension of  $\mathcal{C}$ , denoted by  $\mathcal{C}_L$ , defined by

$$\mathcal{C}_L \phi = \lim_{\varepsilon \rightarrow 0} \mathcal{C} \frac{1}{\varepsilon} \int_0^\varepsilon S_t^* \phi \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \phi(\xi) \, d\xi,$$

with  $\mathcal{D}(\mathcal{C}_L)$  being the set of all  $\phi \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  for which the above limit exists. We refer to [16] for more information on the concept of Lebesgue extension. The output function of the signal generator is  $u(t) = \mathcal{C}_L d_t$ , which is defined for almost every  $t \geq 0$ . It turns out that  $u = d_0$  (the generated signal is the initial state).

We define an extended system **L** by connecting **P** to the generator for  $d_0$  as shown in Figure 1. Then we have

$$\dot{x}(t) = f(x(t), u(t)), \quad (14)$$

$$d_t = S_t^* d_0, \quad (15)$$

$$u(t) = \mathcal{C}_L d_t, \quad (16)$$

$$y(t) = h(x(t)). \quad (17)$$

Let  $z(t) = \begin{bmatrix} x(t) \\ d_t \end{bmatrix}$  denote the state at time  $t$  of the above system, so that  $z(t) \in Z = \mathbb{R}^n \times X$ .

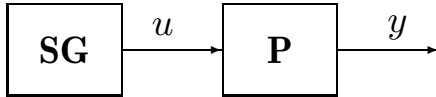


Fig. 1. The extended closed-loop system. The block **SG** is the infinite-dimensional linear signal generator for the  $L^p$  signal  $u$ .

Consider the storage function  $H_{cl} : Z \rightarrow \mathbb{R}_+$  defined for  $\begin{bmatrix} x \\ d \end{bmatrix}$  by  $H_{cl}(z) = H(x) + 2\|d\|^p$ . It can be shown that the function  $H_{cl}$  is absolutely continuous as a function of  $t$ . Since  $H$  is continuous and the solution  $x$  of (14) is absolutely continuous as a function of  $t$  defined in  $\mathcal{J}$ , it follows that  $H(x(t))$  is absolutely continuous on  $\mathcal{J}$ . From (13) and since  $d_0 \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ , it follows that  $\frac{d}{dt} \|d_t\|_{L^p}^p \in L^1(\mathbb{R}_+, \mathbb{R}^m)$ . This implies that  $\|d_t\|_{L^p}^p$  is absolutely continuous on  $\mathbb{R}_+$ .

Using (8), (13), (14) – (16), we obtain that, for almost every  $t \in \mathcal{J}$ ,

$$\begin{aligned} \dot{H}_{cl} &= \frac{\partial H(x)}{\partial x} f(x, u(t)) - 2\|u(t)\|^p \\ &\leq -k\|y(t)\|^p + \|u(t)\|^p - 2\|u(t)\|^p, \\ &\leq -k\|y(t)\|^p - \|u(t)\|^p. \end{aligned} \quad (18)$$

Let us prove that  $\mathcal{J} = \mathbb{R}_+$ . If the maximal interval of definition  $I(x(0))$  is finite, it follows from Corollary 2.3 that  $x(t)$  must leave any compact set  $K \subset \mathbb{R}^n$  at some finite time  $T < I(x(0))$ . Since  $H_{cl}$  is absolutely continuous as a function of  $t$  and bounded from below, (18) implies that  $H_{cl}(z(t))$  is bounded and non-increasing for all  $t \in \mathcal{J}$ . In particular, the state  $x(t)$  never leaves the compact set  $\{x \in \mathbb{R}^n \mid H(x) \leq H_{cl}(z(0))\}$  for all  $t \in \mathcal{J}$ . Hence, we conclude that  $\mathcal{J} = \mathbb{R}_+$ ,  $x(t)$  is bounded for all  $t \geq 0$  and  $H_{cl}(z(t))$  has a limit  $h$  as  $t \rightarrow \infty$ .

We will prove the relative compactness of  $z(\mathbb{R}_+)$ . It has been shown that  $x(t)$  is bounded for all  $t \in \mathbb{R}_+$ , hence  $x(\mathbb{R}_+)$  is relatively compact in  $\mathbb{R}^n$ . Since  $\lim_{t \rightarrow \infty} \|d_t\|_{L^p} = 0$ . This implies that  $\{d_t \mid t \geq 0\}$  is relatively compact in  $L^p(\mathbb{R}_+, \mathbb{R}^m)$ . Therefore  $z(\mathbb{R}_+)$  is relatively compact in  $\mathbb{R}^n \times X$ .

The final proof is to use Proposition 2.4 by showing that  $\Omega(z) = \{0\}$ . Let  $\pi$  denote the semiflow of (14)–(15) so that  $z(t) = \pi(t, z_0)$ . Using Proposition 2.4 and the relative compactness of  $z(\mathbb{R}_+)$ ,  $\Omega(z)$  is non-empty, compact and  $\pi$ -invariant.

For any  $\xi \in \Omega(z)$ , there is a sequence  $(t_n) \in \mathbb{R}_+$  such that  $t_n \rightarrow \infty$  and  $z(t_n) \rightarrow \xi$ . By continuity of  $H_{cl}(z)$ ,  $H_{cl}(\xi) = \lim_{n \rightarrow \infty} H_{cl}(z(t_n)) = h$ . Therefore,  $H_{cl}(z(t)) = h$  on  $\Omega(z)$ . Since  $\Omega(z)$  is  $\pi$ -invariant,  $\Omega(z) \subset E = \{z \in Z \mid \dot{H}_{cl}(z) = 0\}$ .

Let  $M$  be the largest  $\pi$ -invariant set contained in  $E$ . Since  $\Omega(z)$  is  $\pi$ -invariant and  $\Omega(z) \subset E$ , we have  $\Omega(z) \subset M \subset E$ .

In the invariant set  $M$ ,  $H_{cl}$  is constant along state trajectories and  $u = 0$  and  $y = 0$  along such trajectories. By the assumptions of the theorem, the system described by (14) is zero-state detectable i.e.,  $u(t) = 0$  and  $y(t) = 0$  for all  $t \geq 0$  implies that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Also, if  $u(t) = 0$  for all  $t \in \mathbb{R}_+$  then  $d_0 = 0$ , so that  $d_t = 0$  for all  $t \in \mathbb{R}_+$ . Hence, in the invariant set  $M$ ,  $H_{cl}(z) = H_{cl}(0) = 0$  for all  $z \in M$ . Since  $H_{cl}(z) > 0$  for all  $z \neq 0$ , we obtain that  $M = \{0\}$ , hence  $\Omega(z) = \{0\}$ . In particular, using (7) it follows that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

The above argument is true for any initial conditions  $x(0) \in \mathbb{R}^n$  and for any  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ .  $\square$

**Corollary 3.3:** Let the plant **P** be as in Theorem 3.2. Then for every  $x(0) \in \mathbb{R}^n$  there exists a unique solution of (1) with  $u \in L_{loc}^p(\mathbb{R}_+, \mathbb{R}^m)$  in  $\mathbb{R}_+$ .

**Remark 3.4:** For the system **P** satisfying all the assumptions in Theorem 3.2, it has been shown that for every  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  and every initial state  $x(0) \in \mathbb{R}^n$ , there exists solution  $x(t)$  of (1) on  $\mathbb{R}_+$ . It is easy to see from (8) that  $y(t)$  is also in  $L^p(\mathbb{R}_+, \mathbb{R}^m)$  and it satisfies  $\|y_T\|_{L^p}^p \leq \frac{1}{k} \|u_T\|_{L^p}^p + \frac{1}{k} H(x(0))$ . Therefore, the system **P** as in Theorem 3.2 is  $L^p$ -stable and it has a finite  $L^p$ -gain (see Vidyasagar

[15] or van der Schaft [9] for details). It will be shown in Example 5.2 and Corollary 5.3 below that the Assumptions (A1) and (A2) are sufficient conditions for  $\mathbf{P}$  being  $L^p$ -stable.  $\square$

#### IV. STATE CONVERGENCE FOR ISS SYSTEM

The ISS property implies that there exists a smooth  $H : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and functions  $\alpha_i \in \mathcal{K}_\infty$ ,  $i = 1, 2, 3$  and  $\sigma \in \mathcal{K}$  such that

$$\alpha_1(\|x\|) \leq H(x) \leq \alpha_2(\|x\|), \quad (19)$$

$$\frac{\partial H(x)}{\partial x} f(x, u) \leq \sigma(\|u\|) - \alpha_3(\|x\|), \quad (20)$$

for all  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  (see also [12]). Note that  $H$  satisfying (19) is proper.

The result in this section is a result from Theorem 3.2.

**Proposition 4.1:** Suppose that  $p \in [1, \infty)$  and let the plant  $\mathbf{P}$  defined by (1), (2) be ISS so that there exists a proper smooth Lyapunov function  $H : \mathbb{R}^n \rightarrow \mathbb{R}_+$  and functions  $\alpha_i \in \mathcal{K}_\infty$ ,  $i = 1, 2, 3$  and  $\sigma \in \mathcal{K}$  such that (19) and (20) hold. Assume that (A1)-(A2) holds. Suppose that there exists  $k > 0$  such that

$$k\|a\|^p \geq \sigma(\|a\|) \quad \forall a \in \mathbb{R}^m. \quad (21)$$

Then for every initial condition  $x(0) \in \mathbb{R}^n$  and for every  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ , there exists a unique solution  $x$  of (1) on  $\mathbb{R}_+$  and  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$  (and hence  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ ).

*Proof:* The proof is similar to that of Theorem 3.2 where  $H_{cl} : Z \rightarrow \mathbb{R}_+$  is taken to be  $H_{cl}(z) = H(x) + (k + 1)\|d\|^p$  and by noticing that in the invariant set  $M$ ,  $H_{cl}$  is constant along state trajectories and  $x = 0$  and  $d_t = 0$  along such trajectories. Therefore  $\Omega(z) = \{0\}$  and using (7) it follows that  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

The above argument is true for any initial conditions  $x(0) \in \mathbb{R}^n$  and for any  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ .  $\square$

**Remark 4.2:** The alternative proof of Proposition 4.1 can also be given by using Barbălat's lemma. Indeed, using the same approach as in the proof of Theorem 3.2, we can show that for any  $x(0) \in \mathbb{R}^n$ , there exists  $I(x(0)) > 0$  and a unique solution  $x(t)$  of (1) with input  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  on  $\mathcal{J} = [0, I(x(0)))$ .

Using the smooth ISS Lyapunov function  $H$  as assumed in the Proposition 4.1, (20) and (21), we have that

$$\begin{aligned} \dot{H} &= \frac{\partial H(x)}{\partial x} f(x, u) \\ &\leq \sigma(\|u\|) - \alpha_3(\|x\|), \end{aligned} \quad (22)$$

$$\leq -\alpha_3(\|x\|) + k\|u\|^p. \quad (23)$$

Let us prove that  $\mathcal{J} = \mathbb{R}_+$ . Since  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ , integrating (23) from  $t = 0$  to  $T$  we have that  $H(x(T)) \leq H(x(0)) + \|u\|_{L^p} < \infty$  for all  $T \in \mathcal{J}$ . This implies that  $x(t)$  remains in a compact set  $G = \{x \in \mathbb{R}^n \mid H(x) \leq H(x(0)) + \|u\|_{L^p}\}$  for all  $t \in \mathcal{J}$  since  $H$  is proper. Hence, we conclude that  $\mathcal{J} = \mathbb{R}_+$  and  $x \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ . From (23) we also obtain that  $\alpha_3(\|x\|) \in L^1(\mathbb{R}_+, \mathbb{R})$ .

Note that Assumption (A2) implies also that for the compact  $G$ , there exists  $c_5, c_6 > 0$  such that  $\|f(x, u)\| \leq c_5 + c_6\|u\|^p$ , for all  $x \in G$  and  $u \in \mathbb{R}^m$ .

It can be shown that  $\dot{x}$  is *uniformly locally integrable*, i.e., for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $t \geq 0$ ,  $\int_t^{t+\delta} \|\dot{x}(\tau)\| d\tau \leq \varepsilon$ . Indeed, take any  $\varepsilon > 0$  and let  $c = \|u\|_{L^p}^p$ , then by choosing  $\delta < \varepsilon/(c_5 + c_6 c)$  we have

$$\begin{aligned} \int_t^{t+\delta} \|\dot{x}(\tau)\| d\tau &= \int_t^{t+\delta} \|f(x(\tau), u(\tau))\| d\tau \\ &\leq \int_t^{t+\delta} c_5 + c_6\|u(\tau)\|^p d\tau \\ &\leq \int_t^{t+\delta} c_5 + c_6 c d\tau \\ &= (c_5 + c_6 c)\delta < \varepsilon, \end{aligned}$$

for all  $t \geq 0$ .

Since  $x$  is absolutely continuous and  $\dot{x}$  is uniformly locally integrable, it implies that  $x$  is uniformly continuous. In particular,  $\alpha_3(\|x\|)$  is also uniformly continuous. By Barbălat's lemma,  $\alpha_3(\|x\|) \in L^1(\mathbb{R}_+, \mathbb{R})$  and  $\alpha_3(\|x\|)$  is uniformly continuous imply that  $\lim_{t \rightarrow \infty} \alpha_3(\|x(t)\|) = 0$ . By the continuity of  $\alpha_3$  we conclude that  $\lim_{t \rightarrow \infty} \|x(t)\| = 0$ .  $\square$

#### V. EXAMPLES

Note that for system  $\mathbf{P}$  satisfying the assumptions in Theorem 3.2, the convergence result may not be obtained by using Barbălat's lemma or by using [8, Theorem 4.2] even though we can conclude the bounded global solution of the state trajectory  $x$ .

**Example 5.1:** Let the system  $\mathbf{P}$  be described by

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x, u) \quad (24)$$

$$= \begin{bmatrix} -x_1(1 + 1/(u^p + 1) + u^p) - x_2^{p-1} + u \\ x_1^{p-1} \end{bmatrix}, \quad (25)$$

$$y = x_1, \quad (26)$$

where  $p \in \{2n \mid n \in \mathbb{N}\}$ . It can be checked that for each compact set  $K \in \mathbb{R}^2$ , there exists  $a_1, a_2, a_3 > 0$  such that  $\|f(x, u) - f(x, 0)\| \leq a_1 + a_2\|u\| + a_3\|u\|^p$  for all  $x \in K$ . Hence, it does not satisfy the Lipschitz condition in [8].

It can be shown that for each compact set  $K \in \mathbb{R}^2$ , there exists  $c_1, c_2 > 0$  such that

$$\|f(a, u) - f(b, u)\| \leq (c_1 + c_2\|u\|^p)\|a - b\|,$$

holds for all  $u \in \mathbb{R}$  and  $a, b \in K$ , i.e., Assumption (A1) holds. Also, for each  $a \in \mathbb{R}^2$ , there exists  $c_3, c_4 > 0$  such that  $\|f(a, u)\| \leq c_3 + c_4\|u\|^p$  for all  $u \in \mathbb{R}$ , i.e., Assumption (A2) holds. Using a proper storage function  $H(x) = x_1^p + x_2^p$ , it can be checked that by using Young's inequality

$$\begin{aligned} \dot{H} &= -px_1^p(1 + 1/(u^p + 1) + u^p) + px_1^{p-1}u, \\ &\leq -px_1^p + (p-1)x_1^p + u^p = -\|y\|^p + \|u\|^p. \end{aligned}$$

This shows that  $\mathbf{P}$  is dissipative with respect to supply rate  $s(y, u) = \|u\|^p - \|y\|^p$ . The system  $\mathbf{P}$  is also zero-state

detectable. Based on these informations, the application of Barbălat's lemma can only show that  $\lim_{t \rightarrow \infty} \|y(t)\| = 0$  (see, for example, Teel [14]). It follows from Theorem 3.2 that for any  $L^p$  input  $u$ , there exists a unique solution  $x(t)$  defined for all  $t \geq 0$  and  $\lim_{t \rightarrow \infty} x(t) = 0$ .  $\square$

*Example 5.2:* Consider the following single-input single-output plant  $\mathbf{P}$

$$\dot{x} = -xu^{2c} - x + u, \quad y = x, \quad (27)$$

where  $c$  is a positive integer. Using the storage function  $H(x) = \frac{1}{2}x^2$ , we have

$$\begin{aligned} \dot{H} &= -x^2u^{2c} - x^2 + xu \\ &\leq \langle y, u \rangle - \|y\|^2. \end{aligned}$$

From this inequality, it can be shown that  $\mathbf{P}$  has a finite  $L^2$  gain of 1, i.e.,  $\|y_T\|_{L^2} \leq \|u_T\|_{L^2} + \sqrt{2H(x(0))}$  (see Lemma 6.5 in [5] for details).

However, this does not imply that for every  $u \in L^2(\mathbb{R}_+, \mathbb{R})$  the solution  $x(t)$  of (27) exists for  $t \in [0, T)$ ,  $T > 0$  (hence the output function  $y$  is not well defined). Suppose that  $u \in L^2$  is given by

$$u(t) = \begin{cases} t^{-0.25} & t \in [0, 1) \\ t^{-1} & t \in [1, \infty) \end{cases}, \quad (28)$$

so that  $u \in L^2(\mathbb{R}_+, \mathbb{R})$ . Now the state equation (27) can be written as follows:

$$\dot{x} = -xt^{-0.5c} - x + t^{-0.25} \quad \forall t \in [0, 1). \quad (29)$$

It can be checked that the solution of (29) for  $x(0) = 0$  is given by

$$x(t) = x_1(t) \int_0^t x_2(\tau) \tau^{-0.25} d\tau, \quad (30)$$

where

$$x_1(t) = \exp\left(-\frac{1}{1-0.5c}t^{1-0.5c} - t\right)$$

and

$$x_2(\tau) = \exp\left(\frac{1}{1-0.5c}\tau^{1-0.5c} + \tau\right).$$

It can be seen from (30) that the plant  $\mathbf{P}$  as in (27) with input  $u$  as in (28) does not have a solution on any interval of the type  $[0, \delta)$  when  $c = 2$ . It has a unique solution when  $c = 1$  which can also be concluded from Theorem 3.2 with  $p = 2$  since it satisfies Assumption (A1) and (A2). If  $c \geq 2$ , we can always find an  $L^2$  input  $u$  such that (27) does not have a solution  $x(t)$  for  $t \in [0, T)$ ,  $T > 0$ , for example,  $u(t) = t^{-\frac{1}{2c}}$ ,  $t \in [0, 1)$ .  $\square$

Example 5.2 shows that Assumptions (A1) and (A2) are sufficient conditions for systems to be an  $L^p$ -stable. This statement is summarized in the following corollary.

*Corollary 5.3:* Let the plant  $\mathbf{P}$  be defined by (1) and satisfies (A1)-(A2). Assume that  $\mathbf{P}$  has a storage function  $H$  such that it is dissipative with respect to supply rate

$s(y, u) = \|u\|^p - k\|y\|^p$ ,  $k > 0$  and  $p \in [1, \infty)$ . Suppose that  $H(x) > 0$  for  $x \neq 0$ ,  $H(0) = 0$  and  $H$  is proper.

Then  $\mathbf{P}$  is  $L^p$ -stable, i.e., for every  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$  and  $x(0) \in \mathbb{R}^n$ , there exists a unique solution  $x$  of (1) on  $\mathbb{R}_+$  and the output function  $y$  is in  $L^p(\mathbb{R}_+, \mathbb{R}^m)$ . Moreover, the state trajectory  $x$  is bounded.

*Proof:* Let  $u \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ . It follows from the first part of the proof in Theorem 3.2 that for any initial conditions  $x(0) \in \mathbb{R}^n$ , there exists a global solution  $x$  of (1) and the state trajectory  $x$  is bounded.

By the dissipativity of  $\mathbf{P}$ , it is easy to show that

$$\|y\|_{L^p} \leq \left(\frac{1}{k}\right)^{\frac{1}{p}} \|u\|_{L^p} + \left(\frac{1}{k}H(x(0))\right)^{\frac{1}{p}}.$$

Thus  $y \in L^p(\mathbb{R}_+, \mathbb{R}^m)$ .  $\square$

Note that we do not need the zero-state detectability in Corollary 5.3 to conclude the  $L^p$ -stability. If the plant  $\mathbf{P}$  as in Corollary 5.3 is zero-state detectable then it is easy to show that the state trajectory  $x$  converges to zero.

## VI. CONCLUSIONS

We have shown the convergence of the plant state given  $L^p$  input signal for a class of  $L^p$ -stable systems and for a class of ISS systems.

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